8.5 DeMoivre's Theorem

DeMoivre's Theorem is used to raise complex numbers to integer powers.

DeMoivre's Theorem: If $z = a + bi$ is any complex number with trig form $r \text{ cis}\,\theta$ and n is any positive integer then the n^{th} power of z is given by

$$z^n = r^n \text{ cis } n\theta = r^n(\cos n\theta + i \sin n\theta).$$

Example: Find $[2(\cos 15° + i \sin 15°)]^6$. Write the result in standard form.

$$\begin{aligned}[2(\cos 15° + i \sin 15°)]^6 &= (2 \text{ cis } 15°)^6 \\ &= 2^6 \text{ cis } 6(15°) \\ &= 64 \text{ cis } 90° \\ &= 64(\cos 90 + i \sin 90) \\ &= 64(0 + 1 * i) \\ &= 64\, i.\end{aligned}$$

Theorem: If $z = r \text{ cis}\,\theta$ and n is a positive integer then there are n- distinct n^{th} roots of z
They are given by

$$z^{1/n} = r^{1/n} [\text{cis } (\theta + k\,360)/n], \quad k = 0, 1. 2, \ldots, n-1.$$

Example: Find the cube roots of 27, in standard form and graph.

First write 27 in trig form.

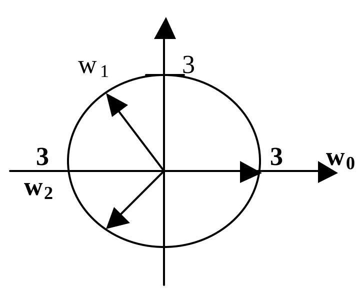
$27 = 27 \text{ cis } \theta = 27(\cos 0 + i\sin 0)$.
So that
$27^{1/3} = 27^{1/3} [\text{cis } (0 + 360\,k)/3]$, $k = 0, 1, 2$.
If we denote the roots as w_k then

$w_0 = 27^{1/3} \text{ cis}[(0 + 360*0)/3] = 27^{1/3} = 3 + 0i$

$$\begin{aligned} w_1 &= 27^{1/3} \text{ cis}[0 + 360*1)/3], \quad k = 1 \\ &= 27^{1/3} \text{ cis } (360/3) \\ &= 3 \text{ cis } 120°, \qquad \text{ref angle is } 60° \\ &= 3(-\cos 60 + i\sin 60) \\ &= 3(-½ + \sqrt{3}\, i/2) \end{aligned}$$

$$\begin{aligned} w_2 &= 27^{1/3} \text{ cis } [(0 + 360 * 2)/3], \quad k = 2 \\ &= 3 \text{ cis } 720/3 \\ &= 3 \text{ cis } (240°) \\ &= -3/2 - 3\sqrt{3}\, i/2 \end{aligned}$$

Converting all reals to decimal form, we have the roots w_1, w_2, and w_3 as
$3 + 0i$, $-1.5 + 2.6\,i$, and $-1.5 - 2.6\,i$



Example: Find the roots of the equation $x^3 + (1 + i\sqrt{3}) = 0$.

Recall any polynomial of degree three has three complex roots.
Now

$x^3 + (1 + i\sqrt{3}) = 0 \longrightarrow x^3 = -(1 + i\sqrt{3})$

$x = \sqrt[3]{-(1 + i\sqrt{3})}$

$-1 - i\sqrt{3} \longrightarrow a = -1,\ b = -\sqrt{3},\ r = \sqrt{1 + 3} = 2.$

So $\theta \in$ Q III.

The ref. angle $\theta' = \tan^{-1}|-\sqrt{3}/-1|$

$= 60°$

Thus $\theta = 180 - 60 = 240$.

Thus $-1 - i\sqrt{3} = 2$ cis $240°$ so that the roots are w_k, where $k = 0, 1, 2$.

$w_0 = 2^{1/3}$ cis $[(240 + 360 * 0)/3] = 2^{1/3}$ cis $240/3 = 2^{1/3}$ cis $80°$

$w_1 = 2^{1/3}$ cis $[(240 + 360 * 1)/3] = 2^{1/3}$ cis $600/3 = 2^{1/3}$ cis $200°$

$w_2 = 2^{1/3}$ cis $[(240 + 360 * 2)/3] = 2^{1/3}$ cis $960/3 = 2^{1/3}$ cis $320°$

HW. p 518 ff. # 1 – 13 (odds), 15, 19, 22, 29.